**Lecture-12 Transformation of E&M field**

1. **Let us consider two plates** in the xz-plane in Frame $S$. The surface charge density $\pm σ$



 $E\_{y}=\frac{σ}{ϵ\_{0}} \left(SI\right)$ ( $E\_{y}=4πσ \left(CGS\right)$ )

The two plates move along x-direction at the speed $v\_{0}$

* $ j\_{x}=σv\_{0}$, thus$ B\_{z}=μ\_{0}j\_{x}=μ\_{0}σv\_{0} \left(SI\right)$ ($B\_{z}=\frac{4π}{c}σv\_{0} (CGS)$)

We consider another frame $S’$, which moves at the speed $u$ along the x-axis with respect to S, what the field observed in $S’$?

In $S’$, the velocity of the two plates:

$v\_{0}^{'}=\frac{v\_{0}-u}{1-v\_{0}u/c^{2}}=c\frac{β\_{0}-β}{1-β\_{0}β}$ here $β\_{0}=\frac{v\_{0}}{c}$ and $β=\frac{u}{c}$

In $S’$, the charge density $σ^{'}=σ\frac{γ\_{0}^{'}}{γ\_{0}}$ (See Lec. 9, Sec. 6)

 here $γ\_{0}=\frac{1}{\sqrt{1-β\_{0}^{2}}}$ and $γ\_{0}^{'}=\frac{1}{\sqrt{1-\left(\frac{β\_{0}-β}{1-β\_{0}β}\right)^{2}}}=\frac{1-β\_{0}β}{\sqrt{(1-β\_{0}^{2})(1-β^{2})}}$

* $σ^{'}=σ\frac{1-β\_{0}β}{\sqrt{((1-β^{2})}}=σγ(1-β\_{0}β)$ here $γ=\frac{1}{\sqrt{1-β^{2}}}$

Thus $j\_{x}^{'}= σ^{'}v\_{0}^{'}= γσ\left(1-β\_{0}β\right)∙ c\frac{β\_{0}-β}{1-β\_{0}β}=σc γ\left(β\_{0}-β\right)$

$$E\_{y}^{'}=\frac{σ^{'}}{ϵ\_{0}}=\frac{σ}{ϵ\_{0}}γ(1-β\_{0}β)$$

* $B\_{z}^{'}=μ\_{0}j\_{x}^{'}=μ\_{0}σ^{'}v\_{0}^{'}=μ\_{0}σcγ\left(β\_{0}-β\right)=μ\_{0}γ(σv\_{0}-σu)$
* $\left\{\begin{array}{c}E\_{y}^{'}=γ\left(E\_{y}-cβB\_{z}\right)\\B\_{z}^{'}=γ(-\frac{β}{c}E\_{y}+B\_{z})\end{array}\right.$ ------ $SI$

We can derive the rules for other components: $S’$ is moving at speed of $v$ along the x-direction with respect to $S$:

$\begin{array}{c}\&E\_{x}^{'}=E\_{x},    E\_{y}^{'}=γ(E\_{y}-cβB\_{z}),    E\_{z}^{'}=γ(E\_{z}+cβB\_{y})\\\&B\_{x}^{'}=B\_{x}, B\_{y}^{'}=γ\left(\frac{β}{c}E\_{z}+B\_{y}\right) , B\_{z}^{'}=γ(-\frac{β}{c}E\_{y}+B\_{z})\end{array}$ ------ $SI$

Or $\begin{array}{c}\&E\_{x}^{'}=E\_{x},    E\_{y}^{'}=γ(E\_{y}-βB\_{z}),    E\_{z}^{'}=γ(E\_{z}+βB\_{y})\\\&B\_{x}^{'}=B\_{x}, B\_{y}^{'}=γ\left(B\_{y}+βE\_{z}\right) , B\_{z}^{'}=γ(B\_{z}-βE\_{y})\end{array}$ -------CGS

Case 1: To the first order ($γ=1$, i.e. $u\ll c$) 🡪 $\rightharpoonaccent{E}^{'}≈\rightharpoonaccent{E}+\rightharpoonaccent{u}×\rightharpoonaccent{B}$

$$ \rightharpoonaccent{B}^{'}≈\rightharpoonaccent{B}-\frac{\rightharpoonaccent{u}}{c^{2}}×\rightharpoonaccent{E}$$

Case 2: Suppose in the $S$ with $B=0$, then:

$\begin{array}{c}\&E\_{x}^{'}=E\_{x},    E\_{y}^{'}=γE\_{y},    E\_{z}^{'}=γE\_{z}\\\&B\_{x}^{'}=0, B\_{y}^{'}=γ\frac{β}{c}E\_{z} , B\_{z}^{'}=-γ\frac{β}{c}E\_{y}\end{array}$

🡪 $B\_{x}^{'}=0, B\_{y}^{'}=\frac{β}{c}E\_{z}^{'} , B\_{z}^{'}=-\frac{β}{c}E\_{y}^{'}$

Then $\rightharpoonaccent{B}^{'}=-\frac{\rightharpoonaccent{u}}{c^{2}}×\rightharpoonaccent{E}'$

Similarly, in the $S$ with $E=0$, then in $S'$: $\rightharpoonaccent{E}^{'}=\rightharpoonaccent{u}×\rightharpoonaccent{B}'$

1. **A conducting rod moving in** $B$**-field (**$v\ll c$**)**

In $S$ frame, $E=0, B=B\hat{z},$ the rod is moving along y-direction.

Lorenz force $\rightharpoonaccent{f}=q\rightharpoonaccent{v}×\rightharpoonaccent{B}$

$\rightharpoonaccent{f}$ drives that charge accumulates at ends （right hand rule）.

* $q\rightharpoonaccent{E}=-\rightharpoonaccent{f}$
* $\rightharpoonaccent{E}=-\rightharpoonaccent{v}×\rightharpoonaccent{B}$

$\rightharpoonaccent{E}:$ induced internal field.

In $S:$



Now let us sit the co-moving frame $S'$ ($u=v$) with $S$. For the moment, we neglect the rod, then we will see in the frame $S'$, there exist $\rightharpoonaccent{B}^{'}$ and $\rightharpoonaccent{E}^{'}$:

$$\rightharpoonaccent{E}^{'}=\rightharpoonaccent{v}×\rightharpoonaccent{B}'≈\rightharpoonaccent{v}×\rightharpoonaccent{B}$$

$\rightharpoonaccent{B}^{'}≈\rightharpoonaccent{B}-\frac{\rightharpoonaccent{v}}{c^{2}}×\rightharpoonaccent{E}≈\rightharpoonaccent{B} $ up to $β^{2}$

In $S'$, the rod is at rest, $\rightharpoonaccent{E}^{'}$ induces charge distribution in the rod, which in turn rebalance the field resulting in no electric field inside the rod! Therefore, there is no motion of electric charge!

Total $\rightharpoonaccent{E}$ distribution in $S'$