Homework1

Problem 1. Levi-Civita symbol、 Kronecker symbol and repeated symbol represents summation

Definition of Levi-Civita symbol:

$$\varepsilon_{ijk} = \begin{cases} 1 \quad \text{if} \quad ijk = 123, 231, 312 \\ -1 \quad \text{if} \quad ijk = 132, 213, 321 \\ 0 \quad \text{others} \end{cases}$$

Definition of Kronecker symbol:

$$\delta_{ij} = \begin{cases} 1 \quad \text{if} \quad i = j \\ 0 \quad \text{if} \quad i \neq j \end{cases} \tag{5}$$

Einstein's repeated index summation:

For the summation formula of $\sum_{i=1}^{n} a_i b_i$, the summation symbol can be inconvenient for complex expressions. Therefore, the following conventions are made:

1. When the subscripts of $a_i b_i$ are the same, the Σ symbol is not written, and the sum is automatically implied.
2. Only when repeated indices appear within the same term, automatic summation is applied. It can only be repeated twice; otherwise, an error occurs. The repeated indices are called dummy indices, while the indices appearing once are called free indices.
3. Indices in the product of the summation cannot be the same. For example, $\sum_{i=1}^{n} a_i b_i \sum_{j=1}^{n} c_j d_j$ can be written as $a_i b_i c_j d_j$ to better represent their independence, which is advantageous for Einstein summation notation.

With the basic knowledge above, please prove the following identity

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$$
$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \quad \text{it also can be written as} \quad \varepsilon_{kij}\varepsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$$

where the repeated symbol represents summation.

Problem 2. Vector algebra

1. Prove the following relations.
   $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$

   $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

   $(\vec{A} \times \vec{B}) \times \vec{C} = -\vec{A}(\vec{B} \cdot \vec{C}) + \vec{B}(\vec{A} \cdot \vec{C})$

   $(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$
2. What are the differences and similarities between a 3-dimensional vector and a pseudo-vector under spatial transformations? Give a few examples of pseudo- vectors.

Problem 3. The vector calculus

Prove the following identities of vector calculus.

1. $f(\vec{r})$ and $g(\vec{r})$ are two scalar functions. Prove that

$$\begin{aligned} \nabla(fg) &= f\nabla g + (\nabla f)g \\ \nabla(f/g) &= \frac{1}{g}\nabla f - f(\nabla g)/g^2 \end{aligned}$$

2. $\vec{A}(\vec{r})$ and $\vec{B}(\vec{r})$ are vector field functions. Prove that

$$\begin{aligned} \nabla\cdot(f\vec{A}) &= f(\nabla\cdot\vec{A}) + \vec{A}\cdot(\nabla f) \\ \nabla\times(f\vec{A}) &= f(\nabla\times\vec{A}) - \vec{A}\times(\nabla f) \\ \nabla\cdot(\vec{A}\times\vec{B}) &= \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B}) \end{aligned}$$

3. This part is not required, and I list a few formulas for future convenience. If you can prove some, you will get extra credits. They are significantly more complicated. If you are interested, TA will teach you some tricks.

$$\begin{aligned} \nabla\times(\vec{A}\times\vec{B}) &= (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B} + \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A}) \\ \nabla(\vec{A}\cdot\vec{B}) &= (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{A} + \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A}) \\ \nabla\times(\nabla\times\vec{A}) &= \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} \end{aligned}$$

Problem 4. Fundamental theorems of vector calculus

The electric field from a point charge q is:

$$\vec{E}(\vec{r}) = \frac{q}{r^3}\vec{r}$$

1. Show that $\nabla\cdot\vec{E}(\vec{r}) = 0$ at $\vec{r} \neq 0$.

   Calculate the surface integral of the electric flux

   Does this result ($\nabla\cdot\vec{E}(\vec{r}) = 0$ at $\vec{r}\neq 0$) violate the Gauss's theorem?
2. Prove that $\nabla\times\vec{E}(\vec{r}) = 0$:

   and

$$\oint_C \vec{E}\cdot \mathrm{d}\vec{l} = 0$$

   for an arbitrary loop. Prove that this means that $\vec{E}$ can be represented as a gradient of a scalar function $\phi(\vec{r})$. Derive this function which is called the electric potential.

Problem 5. Multi-valued functions

1. In a superconductor, each position r is associated with a complex number $\Delta(\vec{r}) = |\Delta(\vec{r})|e^{i\theta(\vec{r})}$. Prove that for an angle distribution function $\theta(\vec{r})$ we have this formula:

$$\oint \mathrm{d}\vec{r}\cdot\nabla\theta(\vec{r}) = 2n\pi$$

   Where n is an integer.
   then search in the literature to find the physical meaning of n .

2. Based on the Stokes theorem, derive that

$$\nabla \times \nabla\theta(\vec{r}) = ?$$