Lecture 2 electrostatics—charges

Outline

- Charges (electric & magnetic)
- Charge conservation
- Charge quantization
- Charge invariance
- Electrostatic energy (potential energy)
- Madelung constant

1. Electric charges

- Fundamental properties: **two classes of charges -- positive and negative** (same types of charges repel each other, attract the other type).

Contrast with: Gravity, only one kind of gravitational mass, and attraction (every mass attracts every other mass.)

Particle – antiparticle (For every kind of particle in nature, there can exist an antiparticle). If intrinsic quality of the particle has an opposite, the anti-particle has that too. While in a property which admits no opposite, e.g. mass, the antiparticle and particle are exactly alike.

electron – positron
proton-antiproton
opposite charges but same mass

Hydrogen ⟷ **anti-hydrogen**

Matter **annihilation** **anti-matter**

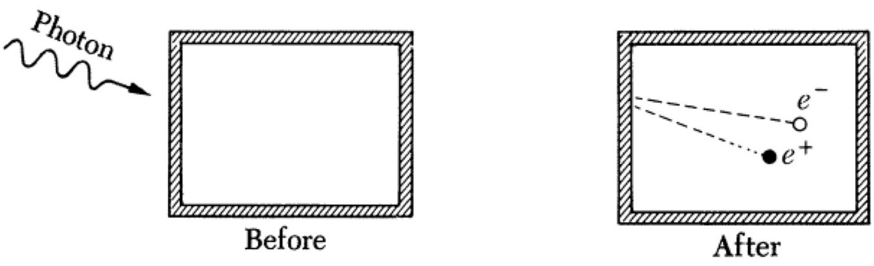
2. Conservation of charge

- **The total electric charge in an isolated system,** that is, the algebraic sum of the positive and negative charge present at any time, **never changes.**

In other words, charge cannot be created or be annihilated. If charge is observed to observed to appear at a certain place, it much come from a place nearby or an opposite charge moves to another place.

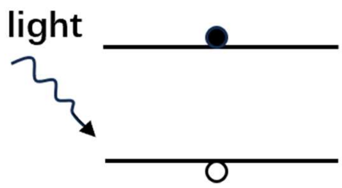
Note the charged particles may vanish or reappear, but they always do so in pairs of equal and opposite charges.

Example 1: A $\gamma - ray$ (a high-energy photon) ends its existence with the creation of a pair of electron and positron in vacuum. They occur simultaneously and then departs.



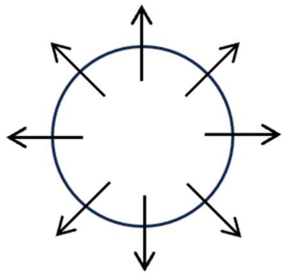
Example 2: **A light generates an electron-hole pair.**

Photovoltaic effect (光伏效应)



The charge conservation law is relativistic invariance.

- **Charge conservation is a local effect!**



$$\frac{\partial Q}{\partial t} + \oiint \vec{j} \cdot d\vec{S} = 0$$ **continuity equation**

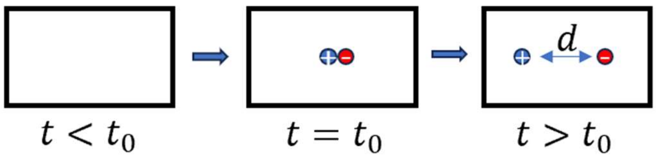
Consider a small volum V, surrounded by a surface S.

The charge within the volume equals to the sum of charge flowing inside and outside. This result applies to an arbitrarily small volume.

$$\iiint_V \frac{\partial \rho}{\partial t} d\tau + \oiint \vec{j} \cdot d\vec{S} = 0 \quad \rightarrow \quad \iiint_V \frac{\partial \rho}{\partial t} d\tau + \nabla \cdot \vec{j}\, d\tau = 0$$

➔ $$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$$ **continuity equation in differential form**

Example 3: If consider a phenomenon that a pair of chrages (+/-) observed created at the same time but at a distance. This cannot happen. They must be created at the same time and place, then depart.



(This is also exact from relativisitic point of view.)

3. Charge invariance

Electric chare is a relativistic invariant quantity. This is far from obvious. In contrast, mass is not invariant under relativistic transformation. Consider a hydrogen molecule, the motion speeds of electrons and proton differ by a factor of the mass ratio. Yet, the hydrogen atom remains charge neutral.

4. Charge quantization

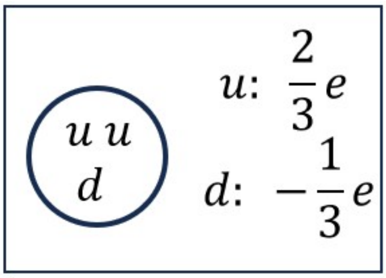
Electric charge is quantized in the unit of e.

Electron carries charge $-e$, and proton carries $+e$.

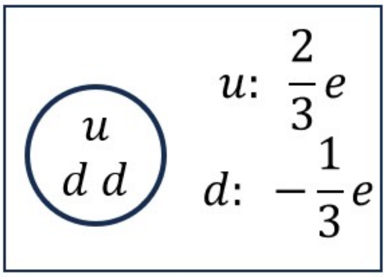
It still a mysterty that why the proton and electron carry exactly opposite charges. However, if their charge differ by a small fraction, then at the macroscopic, matter will not be neutral.

Note: Hadrons, a class that includes the proton and the neutron, involves basic units called quarks **(Quantum** chromodynamics High enery physics). Proton and neutron are composed of three quarks with fractional charges $\pm 1/3\,e$ or $\pm 2/3\,e$.

Proton:

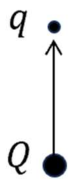


Neutron:



5. Coulomb's law

- **Inverse square law for two stationary point charges:**



$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}\,\hat{r} \quad \text{– SI unit}$$

$$\vec{F} = \frac{Qq}{r^2}\,\hat{r} \quad \text{– CGS unit}$$

$\hat{r}$ shows that the force is parallel to the line joining the charges.

$\epsilon_0 = 8.854 \times 10^{-12}\ C^2/Nm^2$ vacuum permittivity

By symmetry principle and the uniqueness of electrostatics, there is only a unique force by Q at q. The force should be invariant by rotation around the "$Q - q$" line, otherwise the force won't unique.

Actually, electrons can be considered as a point charge ($-e$) and the force between two stationary electrons could be expressed by the Coulomb's law. However, electron also has another intrinsic property: spin. Hence, the interaction between two fixed electrons has the magnetic part in addition to the electrostatic part. Typically, the magnetic dipolar interaction decays as r^{-4}. At about 1 Å, Coulomb force is about 10^4 times the magnetic interaction.

If the charge is moving, it will feel another type of force due to the magnetic field:

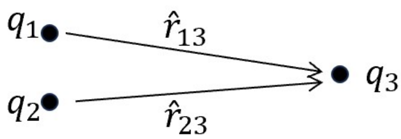
the Lorenz force: $\vec{F} = q\vec{v} \times \vec{B}\ \ (SI)$ or $\vec{F} = q\frac{\vec{v}}{c} \times \vec{B}\ \ (SI)$

Is there a range of distances where it completely breaks down?

Short range limit: less $0.1\ fm$, where the electromagnetic theory breaks down.

Longe range limit: very long (One stringent test $10^8\ m$: the spatial variation of magnetic field in Jupiter was carefully analyzed to be entirely consistent with classical EM theory, done in the mission of Pioneer 1). From geographic to astronomical

- **Superposition law –** electric force and charges are additive.



$\vec{F}_{13} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_3}{r_{13}^2}\hat{r}_{13}$, $\vec{F}_{23} = \frac{1}{4\pi\epsilon_0}\frac{q_2 q_3}{r_{23}^2}\hat{r}_{23}$

Then $\vec{F}_3 = \vec{F}_{13} + \vec{F}_{23} = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_3}{r_{13}^2}\hat{r}_{13} + \frac{q_2 q_3}{r_{23}^2}\hat{r}_{23}\right)$

If $r_{13}, r_{23} \gg r_{12}$, $\vec{F}_3 = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_3 + q_2 q_3}{r_3^2}\hat{r}_3$

The force with which two charges interact is not changed by the presence of a third charge.

Force on q_0 from n other charges, $\vec{F}_0 = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_0 q_i}{r_{i0}^2}\hat{r}_{i0}$.

6. Electric field

- **Definition: $\vec{E} = \vec{F}/q$, for a fixed charge.**

For a point charge Q, $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$

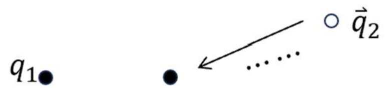
For n charges, $\vec{E} = \vec{F}_0/q = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_{i0}^2}\hat{r}_{i0}$

Unit: [N/C] or [V/m]

7. Energy of charges

- For an electrostatic system, the electric force is a conservative force. Suppose two charges q_1 and q_2 are separated by a distance r_{12}. The potential energy equals to the work done to overcome the electrostatic force to bring two charges from infinity to the distance r.

$$U = \int_{\infty}^{r_{12}} -\vec{F}_{12} \cdot d\vec{r} = \frac{1}{4\pi\epsilon_0} \int_{\infty}^{r_{12}} -\frac{q_1 q_2}{r_{12}^2} \cdot dr = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}}$$



Here "−" mean the force is to overcome the electrostatic force.

- Then, bring the 3rd charge q_3 from infinity to finite distances to q_1, q_2, the work done is:

$$U = \int_{\infty}^{r_{13}} -\vec{F}_{13} \cdot d\vec{r} + \int_{\infty}^{r_{23}} -\vec{F}_{23} \cdot d\vec{r} = \frac{1}{4\pi\epsilon_0} \left(\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$

The potential energy is superposition of the energy between q_1 and q_3 in absence of q_2 and the energy between q_2 and q_3 in absence of q_1.

- The total potential energy:

$$U = \frac{1}{4\pi\epsilon_0} \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$

1) We assume that the potential energy is zero when all changes are separated at infinite distances.
2) U is symmetric with respect to q_1, q_2 and q_3. Hence, it is independent on the way we bring charges together.

- For n-charges, $U = \frac{1}{4\pi\epsilon_0} \frac{1}{2} \sum_{j=1}^{n} \sum_{i \neq j} \frac{q_i q_j}{r_{ij}}$

★ Potential (electrostatic) energy of crystal – Madelung constant

Madelung constant is used in determining the electrostatic potential of a single ion in a crystal by approximating the ions by point charges. It is named after Erwin Madelung, a German physicist.
For simplicity, let us consider a one dimensional crystal,

$$u = \frac{1}{2} e^2 \left(-\frac{1}{a} + \frac{1}{2a} - \frac{1}{3a} + \cdots\right) \times 2 = -\frac{e^2}{a}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right) = -\frac{e^2}{a} ln2$$

The electrostatic energy is negative. The negative sign shows that work would have to be done to take the crystal apart into ions. In other words, the electrical energy helps to explain the cohesion of the crystal. The electrostatic energy is negative, which partly explain the binding energy of material. But electrostatic energy cannot be the whole story, since it does not have a minimum! If let $a \to 0$, the electrostatic energy would go to negative infinity! This is impossible! We need quantum theory to explain the stability of matter, which is due to Pauli's exclusion principle of fermions. Later, we will prove that only by electrostatics, a system consisting of positive and negative charges cannot reach equilibrium.

- **Exercise:** calculate the Madelung constant in a 2D or 3D lattice: See Chap. 1.6 Purcell's book.